

# A mathematical review on the multiple-solution problem\*

K. Zhu,<sup>†</sup> X. H. Mo, C. Z. Yuan, P. Wang

*Institute of High Energy Physics, Chinese Academy of Sciences,*

*P.O. Box 918(1), Beijing 100049, China*

The recent multiple-solution problem in extracting physics information from a fit to the experimental data in high energy physics is reviewed in a mathematical viewpoint. All these multiple solutions were found via a fit process previously, while in this letter we prove that if the sum of two coherent Breit-Wigner functions is used to fit the measured distribution, there should be two and only two non-trivial solutions, and they are related to each other by analytical formulae. For real experimental measurements in more complicated situations, we also provide a numerical method to derive the other solution from the already obtained one. The excellent consistency between the exact solution obtained this way and the fit process justifies the method. From our results it is clear that the physics interpretation should be very different depending on which solution is selected. So we suggest that all the experimental measurements with potential multiple solutions be re-analyzed to find the other solution because the result is not complete if only one solution is reported.

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## 1. Introduction

Interference as a nature phenomenon has been observed for a very long time in situations where waves intersect, no matter the mediate material is water, string, sound, or light. It has been studied in depth and also widely used in a range of physical and engineering measurements. However, classic physics and quantum mechanics provide basically different explanations of this phenomenon. In classic physics, if two meeting waves are considered contributing to a process, what observed is just the sum of amplitudes of two waves, i.e.,  $A(x) = A_1(x) + A_2(x)$ , where  $x$  is a generalized coordinate which could be position, momentum, time, energy, etc. But in quantum mechanics, their wave functions are summed to obtain the total amplitude (generally there are relative phases between them), i.e.  $|\psi(x)\rangle = |a(x)\rangle + |b(x)\rangle$ . And the experimentally measured quantities are usually proportional to the modulus of the amplitude squared, and thus one generally has contribution from an interference

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<sup>†</sup>e-mail: zhuk@ihep.ac.cn

term  $\langle a|b \rangle$ . Compared with the classic ones, many new and fantastic features are caused by this additional interference term  $\langle a|b \rangle$ , and the ambiguity in extracting information from observation is one of them.

Usually the experimental quantities depending on  $|\psi|^2$  are measured, and from which we extract the information of the amplitudes  $|a\rangle$  and  $|b\rangle$ . Unlike in the classic physics case, as there is a square operation between the observable and the amplitudes, we would expect other solutions,  $|a'\rangle$  and  $|b'\rangle$ , be found in extracting amplitudes from physics measurements. It is true that the existing freedom on the global phase is non-relevant to the physics in this extraction procedure. However, more and more experimental analyses presented recently imply that different solutions with different relative phases would lead to non-trivial different physics interpretations.

Some earlier examples reporting multiple solutions are in the study of the so-called  $Y$  states via initial state radiation ( $ISR$ ) by the Belle experiment<sup>1,2</sup>. The invariant mass distributions of  $\pi^+\pi^-J/\psi$  and  $\pi^+\pi^-\psi(2S)$  are fitted with two coherent resonant terms and an incoherent background term. Another example is the study of the decay dynamics of  $\eta' \rightarrow \gamma\pi^+\pi^-$  mode. When the  $\pi^+\pi^-$  invariant mass distribution is fitted with a coherent sum of the  $\rho$  resonance and a contact term, two solutions are found with one solution corresponding to constructive interference between the two amplitudes while the other corresponding to destructive interference.<sup>3</sup> Some recent examples are presented in Refs. 4 and 5. In Ref. 4 two solutions are found for both the branching fraction measurement of  $\phi \rightarrow \omega\pi^0$  and the  $\rho - \omega$  mixing study. In Ref. 5, four sets of solutions are found in fitting the  $R$ -values to extract the resonant parameters of the excited  $\psi$  states, namely the  $\psi(4040)$ ,  $\psi(4160)$ , and  $\psi(4415)$ .

However, it is notable that in Refs. 1–6 all the multiple solutions are found via fitting process. And we know fit method always suffers from background uncertainties and limited statistics. Then some interesting questions are raised naturally such as: whether they are exact solutions or only approximate results due to the shortcomings of the fit process or statistical fluctuation; whether these solutions always exist or just appear in some special cases; how many fold ambiguities there are if multiple-solution exists; and if one special solution has already been found, whether the others can be derived from it. Some of these questions are explored from physics point of view in Refs. 4 and 5, and mathematical attempts are described in Ref. 7 and 8. Following the clue of these above mentioned studies we perform the present investigations in this paper. Comparing with Ref. 7 in which Fourier transformation is applied, the solution finding process is extremely simplified in this study; comparing with Ref. 8, more general conclusions are obtained in this analysis.

In section 2, at the outset, a general and mathematic model for the sum of two amplitudes is established on the basis of the known facts from physical analyses. If two amplitudes are both the commonly used Breit-Wigner (BW) functions, the analytical expression for the two solutions are obtained. Moreover, an effective ap-

proach is developed for deriving the algebra equations related to the solutions. Then many double-solutions are deduced for distinctive forms of amplitude functions. After that, we put forth a constraint on the ratio of the two amplitude functions, which ensures that there will be non-trivial double solutions. In section 3, we use a toy numerical example to check and confirm our results. We also develop a numerical procedure to get the unknown solution from the known one when the form of the amplitude function is extremely complicated. Finally there is a short discussion on the consequence of having multiple solutions in extracting physics information from the experimental data, and we also present our suggestion on how to handle this situation.

## 2. Mathematical methodology

If scrutinizing the relevant analyses with multiple solutions<sup>1–6</sup>, we can note two prominent characteristics: 1. all set of solutions have equal goodness-of-fit; 2. although all parameters including the masses, the total widths, the partial widths and some other related parameters are allowed to float in the fit, it is observed that the differences between multiple solutions are only in the partial widths and the relative phases between the amplitudes. The first point indicates that all solutions are mathematically equivalent while the second point implies that the main difference for different solutions is in the normalization factor and the relative phase between them. In the light of these experimental facts, we abstract a general mathematical model for multi-solution problem. Without losing generality, the study that follows focuses on the case of two amplitude functions.

### 2.1. Solutions for two BW amplitudes

Generally, a sum of two quantum amplitudes can be described by a complex function  $e(x, z_1, z_2)$  with form

$$e(x, z_1, z_2) = z_1 g(x) + z_2 f(x) , \quad (1)$$

where  $g(x)$  and  $f(x)$  are both complex functions,  $x$  is a real variable, and  $z_1, z_2$  are complex numbers. Our goal is to find non-trivial different series of parameters  $z'_1$  and  $z'_2$  that satisfy

$$|e(x, z_1, z_2)|^2 = |e(x, z'_1, z'_2)|^2 . \quad (2)$$

Noticed that the global phase plays no role in the amplitude squared, we can reduce the dimension of  $z_1 - z_2$  parameter space to a  $z - d$  space in which  $d$  is real number, and re-write  $|e(x, z_1, z_2)|^2$  to a more convenient form by defining

$$\begin{aligned} |e(x, z_1, z_2)|^2 &\equiv \frac{1}{d} |g(x) + z f(x)|^2 = \frac{|g(x)|^2}{d} \left| 1 + z \frac{f(x)}{g(x)} \right|^2 \\ &\equiv \frac{|g(x)|^2}{d} |1 + z F(x)|^2 \equiv \frac{|g(x)|^2}{d} E(x, z) . \end{aligned} \quad (3)$$

Here  $F(x) \equiv f(x)/g(x)$  and  $E(x, z) \equiv |1 + z F(x)|^2$ . Since  $|g(x)|^2$  is only a multiply factor and is independent of  $d$  and  $z$ , it can be dropped in the following discussion. Now we only focus on finding different series of  $d$  and  $z$  that keep  $E(x, z)/d$  unchanged. Denoting the real and imaginary parts of  $F(x)$  with  $R_F(x)$  and  $I_F(x)$ , as well as  $R_z$  and  $I_z$  for  $z$ , respectively, and expressing  $E(x, z)$  by these real and imaginary components, we obtain

$$E(x, z) = (R_F^2 + I_F^2)(R_z^2 + I_z^2) - 2I_F I_z + 2R_F R_z + 1. \quad (4)$$

For compactness, the explicit dependence on  $x$  of  $R_F(x)$  and  $I_F(x)$  is removed here. Without losing generality, set  $d = 1$  as an initial solution for convenience, so our task is to find all possible  $d'$  and  $z'$  to render  $E(x, z')/d' = E(x, z)$ . To specialize our work, we consider the case when both  $g(x)$  and  $f(x)$  are non-relativistic BW functions<sup>9</sup>

$$g(x) = \frac{\Gamma_g}{(x - M_g) + i\Gamma_g},$$

$$f(x) = \frac{\Gamma_f}{(x - M_f) + i\Gamma_f},$$

where  $M$  and  $\Gamma$  are the mass and width of a resonance, respectively. This BW-form amplitude function is chosen because it's broadly adopted in high energy physics. With the above forms of  $g(x)$  and  $f(x)$ , the real and imaginary components of  $F(x)$  are

$$R_F = \frac{\Gamma_f[\Gamma_g\Gamma_f + (M_g - x)(M_f - x)]}{\Gamma_g[\Gamma_f^2 + (M_f - x)^2]},$$

and

$$I_F = \frac{\Gamma_f[\Gamma_f(M_g - x) - \Gamma_g(M_f - x)]}{\Gamma_g[\Gamma_f^2 + (M_f - x)^2]}.$$

After some algebra, we get an interesting relation

$$R_F^2 + I_F^2 = aR_F + bI_F + c, \quad (5)$$

with

$$a = \frac{\Gamma_g + \Gamma_f}{\Gamma_g}, \quad b = \frac{M_g - M_f}{\Gamma_g}, \quad c = -\frac{\Gamma_f}{\Gamma_g}. \quad (6)$$

With Eq. (5),  $E(x, z)$  is recast as

$$R_F(aR_z^2 + aI_z^2 + 2R_z) + I_F(bR_z^2 + bI_z^2 - 2I_z) + c(R_z^2 + I_z^2) + 1. \quad (7)$$

Similar expression can be obtained for  $E(x, z')$ . Notice that  $R_F$  and  $I_F$  are functions in variable space ( $x$  space), while  $R_z$  and  $I_z$  are functions in parameter space ( $z - d$  space), if we want  $E(x, z')/d' = E(x, z)$  for any  $x$ , then the corresponding functions

in parameter space (the coefficients of the functions in variable space) should be equal. This requirement immediately yields

$$\begin{aligned} aR_{z'}^2 + aI_{z'}^2 + 2R_{z'} &= d'(aR_z^2 + aI_z^2 + 2R_z) , \\ bR_{z'}^2 + bI_{z'}^2 - 2I_{z'} &= d'(bR_z^2 + bI_z^2 - 2I_z) , \\ cR_{z'}^2 + cI_{z'}^2 + 1 &= d'(cR_z^2 + cI_z^2 + 1). \end{aligned} \quad (8)$$

In the light of the set of equations, it turns out that  $d'$  must satisfy a second order equation and there are two roots of it. One is the trivial solution with  $d' = 1$  and  $z' = z$  correspondingly, and the other one is

$$\begin{aligned} d' &= \frac{a^2 + b^2 + 4c}{(a - 2R_z c)^2 + (b + 2I_z c)^2} \\ z' &= \left( R_z d' - \frac{a(d' - 1)}{2c} \right) + \left( I_z d' + \frac{b(d' - 1)}{2c} \right) i \end{aligned} \quad (9)$$

## 2.2. Some special solutions

In this section we consider some other forms for  $f(x)$  and  $g(x)$ . The first form is from the KLOE experiment on  $e^+e^- \rightarrow \omega\pi^0$  with both  $\omega \rightarrow \pi^+\pi^-\pi^0$  and  $\omega \rightarrow \gamma\pi^0$ . The cross section of  $e^+e^- \rightarrow \omega\pi^0$  in the vicinity of the  $\phi$  resonance, as a function of the center-of-mass energy,  $\sqrt{x}$ , is parameterized as

$$\sigma(\sqrt{x}) = \sigma_{nr}(\sqrt{x}) \cdot \left| 1 - z \frac{M_\phi \Gamma_\phi}{D_\phi(\sqrt{x})} \right|^2 \quad (10)$$

in Ref. 10, where  $\sigma_{nr}(\sqrt{x}) = \sigma_0 + \sigma'(\sqrt{x} - M_\phi)$  is the bare cross section for the non-resonant process, parameterized as a linear function of  $\sqrt{x}$ ;  $M_\phi$ ,  $\Gamma_\phi$ , and  $D_\phi = M_\phi^2 - x - iM_\phi\Gamma_\phi$  are the mass, the width, and the inverse propagator of the  $\phi$  meson, respectively. Here  $z$  is a complex number which depicts the interference effect. Comparing with definition of  $E(x, z)$ ,  $f(x)$  and  $g(x)$  have the forms

$$g(x) = -1, f(x) = \frac{M_\phi \Gamma_\phi}{M_\phi^2 - x - iM_\phi \Gamma_\phi}.$$

The simple algebra yields  $R_F^2 + I_F^2 = -I_F$ , which in turn gives

$$\begin{aligned} R_{z'}^2 + I_{z'}^2 + 2I_{z'} &= d'(R_z^2 + I_z^2 + 2I_z) , \\ R_{z'} &= d'R_z , \\ 1 &= d' . \end{aligned} \quad (11)$$

With the last equality  $d' = 1$ , the relation  $R_{z'} = d'R_z$  implies  $R_{z'} = R_z$ , then the first equation provides the other non-trivial solution

$$z' = R_z - i(I_z + 2) .$$

They are just the results acquired in Ref. 8 by another method.

As the second example, we consider the form

$$g(x) = \frac{1}{x}, \quad f(x) = \frac{1}{m^2 - x + im\Gamma} , \quad (12)$$

which is usually used to extract the resonant information of  $\omega$  in fitting the data of  $e^+e^- \rightarrow \pi^+\pi^-$  cross section<sup>11</sup>. Here  $m$  and  $\Gamma$  indicate the mass and total decay width of the resonance. Accordingly, we obtain  $R_F^2 + I_F^2 = -R_F + \zeta I_F$  ( $\zeta = -m/\Gamma$ ), which in turn gives

$$\begin{aligned} R_{z'}^2 + I_{z'}^2 - 2R_{z'} &= d'(R_z^2 + I_z^2 - 2R_z) , \\ \zeta(R_{z'}^2 + I_{z'}^2) - 2I_{z'} &= d'[\zeta(R_z^2 + I_z^2) - 2I_z] , \\ 1 &= d' . \end{aligned} \quad (13)$$

After some algebra, we get the other non-trivial solution

$$z' = \frac{2 + (\zeta^2 - 1)R_z - 2\zeta I_z}{1 + \zeta^2} + i \frac{2\zeta(1 - R_z) - (\zeta^2 - 1)I_z}{1 + \zeta^2} . \quad (14)$$

We consider a more general case, when  $f(x)$  and  $g(x)$  are any non-trivial functions, but their ratio must ensure that the real or imaginary component of  $F(x)$  is constant<sup>a</sup>. In this case, there exist two solutions. Specially speaking, when  $F(x) = \kappa + ih(x)$ , with  $h(x)$  being a non-trivial function and  $\kappa$  is a non-zero real constant. Here

$$E(x, z) = h^2(x)(R_z^2 + I_z^2) - 2h(x)I_z + \kappa(R_z^2 + I_z^2) + 2\kappa R_z + 1 . \quad (15)$$

From  $E(x, z')/d' = E(x, z)$  and we require all coefficients of each order of  $h(x)$  being equated respectively, i.e.

$$\begin{aligned} R_{z'}^2 + I_{z'}^2 &= d'(R_z^2 + I_z^2) , \\ -2I_{z'} &= d'(-2I_z) , \\ \kappa^2(R_{z'}^2 + I_{z'}^2) + 2\kappa R_{z'} + 1 &= d'[\kappa(R_z^2 + I_z^2) + 2\kappa R_z + 1] . \end{aligned} \quad (16)$$

Besides the trivial solution  $d' = 1$  and  $z' = z$ , there exists the other solution to above set of equations,

$$\begin{aligned} d' &= \frac{1}{4\kappa^2(R_z^2 + I_z^2) + 4\kappa R_z + 1} , \\ z' &= -d'[2\kappa(R_z^2 + I_z^2) + R_z] + i(I_z d') . \end{aligned} \quad (17)$$

When  $F(x) = h(x) + i\kappa$ , via similar derivation, the other non-trivial solution is obtained as

$$\begin{aligned} d' &= \frac{1}{4\kappa^2(R_z^2 + I_z^2) - 4\kappa I_z + 1} , \\ z' &= R_z d' + i d'[2\kappa(R_z^2 + I_z^2) - I_z] . \end{aligned} \quad (18)$$

There are also other cases where there are two solutions, such as when  $R_F$  is a linear function of  $I_F$ , or vice versa, we will not discuss them in details here.

<sup>a</sup>This can be realized, for example, if  $f(x) = \rho_f(x)e^{i\theta_f(x)}$  and  $g(x) = \rho_g(x)e^{i\theta_g(x)}$ , where  $\rho_{f,g}(x)$  and  $\theta_{f,g}(x)$  are any non-trivial functions. As long as there exist relations  $\rho_f(x) = \rho_g(x) \cdot \sqrt{h^2(x) + \kappa^2}$  and  $\theta_f(x) = \theta_g(x) + \tan^{-1}(h(x)/\kappa) + 2\pi n$  ( $n$ : any integer), it always has  $F(x) = \kappa + ih(x)$ .

### 2.3. Constraint on the amplitude functions

Despite of the examples shown in the previous section, it is clear that the double-solution issue is not universal for any forms of the functions. Actually, it is easy to find some forms of  $f(x)$  and  $g(x)$ , in which no multiple-solutions can be found,  $g(x) = x$  and  $f(x) = x^3 + ix^2$  is such an example.

This is why, although still far from the final answer, we want to discuss what kind of constraints can be applied to the functions if double-solutions exists, i.e. what kind of amplitude functions,  $f(x)$  and  $g(x)$  in the preceding section, will guarantee double solutions. When we return to the study of two BW amplitudes, we notice that the relation in Eq. (5) is crucial for obtaining the double-solutions, and this relation provides a constraint on  $F(x)$ . It is easy to check that all the special forms with double-solutions found by us, obey this requirement. In short: there will be two solutions if the Argand diagram of  $F(x)$  is a circle.

### 3. Check and Application

As a cross check, let's consider an ad hoc example: the parameters of the two BW functions and one solution are set as

$$M_g = 3.0, \quad \Gamma_g = 0.4, \quad M_f = 2.1, \quad \Gamma_f = 0.1, \quad z = 1 - i.$$

Using the aforementioned method, we can find another solution, which is exactly repeated by fitting with maximum likelihood method. The comparison of the results is shown in Table 1.

Table 1. Comparison between exact solution and that obtained from fit process in the case of summing up two simple BW functions. For the fit, toy MC is used to generate a 10,000 events data sample.

Item	Input	The other sol.	Fit I	Fit II
$d$	1	0.529	—	—
$R_z$	1	0.647	$1.019 \pm 0.054$	$0.644 \pm 0.040$
$I_z$	-1	1.588	$-1.019 \pm 0.060$	$1.601 \pm 0.028$
$M_g$	3.0	3.0	$3.011 \pm 0.010$	$3.011 \pm 0.010$
$\Gamma_g$	0.4	0.4	$0.402 \pm 0.010$	$0.402 \pm 0.010$
$M_f$	2.1	2.1	$2.101 \pm 0.003$	$2.101 \pm 0.003$
$\Gamma_f$	0.1	0.1	$0.101 \pm 0.003$	$0.101 \pm 0.003$

This example indicates that in principle, the fit procedure can be used as a feasible approach to find the multiple solutions from the experimental data.

It is obvious that, for the two BW amplitudes case, if one solution is obtained by fitting to the data, the other one can be readily and analytically obtained by applying Eq. (9). This definitely saves a lot of time and energy. However, due to the complexity of the expressions in practice, the solution has to be obtained from the

following numerical method. Firstly, we draw  $F(x)$  in the complex plane to check whether it is a circle. If the answer is yes, we need to determine the parameters  $a$ ,  $b$ , and  $c$  in relation  $R_F^2(x) + I_F^2(x) = aR_F(x) + bI_F(x) + c$ . This can be achieved by randomly selecting three points on the curve, i.e. set  $x$  in  $F(x)$  with three randomly chosen values, to obtain three linear equations with  $a$ ,  $b$ , and  $c$  as variables, whose values can be got by solving the set of equations. This process should be realized numerically when  $F(x)$  takes a very complicated form in practice. With  $a$ ,  $b$ , and  $c$  obtained, we can derive the other solution with Eq. (9) as was shown before. We illustrate this method by the examples which are selected from the initial state radiation measurements at BaBar and Belle<sup>1,12</sup>, where the  $\pi^+\pi^-\psi(2S)$  and  $\pi^+\pi^-J/\psi$  invariant mass distributions are described by two coherent resonances. The cross sections are formulated as

$$\sigma(s) = |BW_1(s) + BW_2(s) \cdot e^{i\phi}|^2,$$

where  $BW_1$  and  $BW_2$  represent the two resonances and  $\phi$  is the relative phase between them. Then in our frame  $BW_1$ ,  $BW_2$ , and  $e^{i\phi}$  represents  $g(x)$ ,  $f(x)$ , and  $z$  respectively, and  $F(x) = BW_2/BW_1$ . And the BW form of a single resonance in these two papers is

$$BW(s) = \sqrt{\frac{M^2}{s}} \frac{\sqrt{12\pi\Gamma_{e^+e^-}B(R \rightarrow f)\Gamma_{tot}}}{s - M^2 + iM\Gamma_{tot}} \sqrt{\frac{PS(s)}{PS(M)}}, \quad (19)$$

where  $M$  is the mass of the resonance,  $\Gamma_{tot}$  and  $\Gamma_{e^+e^-}$  are the total width and partial width to  $e^+e^-$ , respectively,  $B(R \rightarrow f)$  is the branching fraction of  $R$  decays into final state  $f$ , and  $PS(s)$  is the three-body decay phase space factor. Noticed here that  $F(x)$  is very complicated. We draw it in the complex plane and find it is in a good circle shape, then we can obtain the parameters  $a$ ,  $b$ , and  $c$  from this plot. Using Eq. (9) and the first solution as input we obtain the second solution as shown above; or reversely, using the second solution as input to obtain the first one. All the results from our method and comparisons with the experimental fits are shown in Table 2, where only the statistical errors are quoted for experimental measurements and only the central values are used as input to get the other solution. From the Table it is clear that our results reproduce the results from the fit process very well, and we consider this as a justification of our method.

#### 4. Discussion

As been found, when the measured distribution is described by  $|g(x) + zf(x)|^2/d$  and  $F(x) = f(x)/g(x)$  fulfills the relation of Eq. (5), i.e.,  $F(x)$  is a circle in complex plane, there are and only are two non-trivial solutions. It has also been proved that if  $f(x)$  and  $g(x)$  are both simple BW functions, this relation is exactly satisfied and Eq. (9) can be utilized to derive the other solution analytically from the one obtained from the fit. For other transmogrified BW functions, which could be extremely complicated, the relation Eq. (5) still hold for  $F(x)$  by numerical checking.



Table 2. Comparison of the exact solutions with those obtained from the fit process for two real experimental measurements. For experimental results, only statistical errors are quoted; for our method, solution I is obtained with solution II as input, and vice versa.

Solutions	I			II		
Items	$B\Gamma_{e^+e^-}(R1)$	$B\Gamma_{e^+e^-}(R2)$	$\phi$	$B\Gamma_{e^+e^-}(R1)$	$B\Gamma_{e^+e^-}(R2)$	$\phi$
fit results in Ref. 1	$5.0 \pm 1.4$	$6.0 \pm 1.2$	$12 \pm 29$	$12.4 \pm 2.4$	$20.6 \pm 2.3$	$-111 \pm 7$
by our method	4.7	6.4	12	12.8	20.4	-111
fit results in Ref. 12	$11.1^{+1.3}_{-1.2}$	$2.2^{+0.7}_{-0.6}$	$18^{+23}_{-24}$	$12.3 \pm 1.2$	$5.9 \pm 1.6$	$-74^{+16}_{-12}$
by our method	11.1	2.1	18	12.3	6.0	-74

So there will be double solutions for these forms too and with  $a$ ,  $b$ , and  $c$  obtained numerically the other solution can be derived by using the same method. The excellent consistency between our exact solutions and experimental fit results justifies this method. We should mention that when we use the experimental fit results as input, only the central values are considered to prove the method is valid. In principle, one can obtain the errors of the parameters in the second solution given the full covariant matrix is available for the first solution (unfortunately, very often, experimental papers only give diagonal errors, and the correlations between the variables are not reported). The uncertainty of the numerical method is not discussed in case of complicated BW forms, as the main purpose of this paper is to examine whether the other solution exist or not, and how to find it.

We also notice that for both solutions, the parameters of each resonance are same but the normalization factors. This implies that the couplings to decay channels are different for different solutions and some experimental reports may not be complete if only one solution was reported while there are two-fold ambiguities. So we suggest any experiment measurement with potential multiple-solution problem redo the analysis to find out the other solutions. Finally, we should point out that from Eq. (4) we may find more conditions where double solutions exist. For example, if the real or virtual component of  $F(x)$  is zero or the real component of  $F(x)$  is a linear function of the virtual one, there should be double solutions too. However, they are not normal in high energy physics so we do not discuss them in detail here. Furthermore, only the sum of two coherent amplitudes is considered in this paper, the generalization to more amplitudes is still in progress.

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